Journal of Engineering Physics and Thermophysics, Vol. 79, No. 6, 2006

DETERMINATION OF THE NONLINEAR HEAT-CONDUCTIVITY COEFFICIENT OF TUBULAR BODIES BY THE METHOD OF FUNCTIONAL IDENTIFICATION

V. T. Borukhov,[a] V. I. Timoshpol'skii,[b] G. M. Zayats,[a] E. V. Kalinevich,[c] and V. A. Tsurko[a]

UDC 519.6

The problem on determination of the nonlinear heat-conductivity coefficient of a body in the cylindrical coordinate system by the method of functional identification has been considered. Computational experiments are described and calculation results are discussed.

Introduction. At present, gradient methods of minimization of goal functionals are being used widely in the numerical solution of inverse problems of mathematical physics. For example, an efficient method is the conjugate-gradient method [1, 2] involving regulating procedures. This method, when used for determining the nonlinear heat-conductivity coefficient $\lambda(T)$ of a body, allows one to calculate the operator conjugated to the internal-superposition operator. However, the standard approach to such calculations involves the replacement of variables in the double integral, which complicates the computational procedure and makes its control difficult. Because of this, the finite-dimension approximation of desired coefficients was performed in many works (see, e.g., [2–6]); thus, the initial functional infinite-dimension problem was transformed into a finite-dimension one. This approach has both advantages, associated with the regulating capabilities of finite-dimension approximations, and disadvantages that are due to the a priori uncertainty of the basis approximation functions selected and their number.

In [7], it was proposed to perform the functional identificaiton of the heat-conductivity coefficient of a body by gradient minimization of the goal functional without a preliminary finite-dimension approximation of the desired nonlinear coefficient. This became possible due to the integro-differential representations of the operator conjugated to the internal-superposition operator, obtained in [7], which do not involve the replacement of variables in the double integrals. Different variants of functional identification of the heat-conductivity coefficient $\lambda(T)$ of a body in the Cartesian coordinate system with the use of a nonlinear parabolic equation were considered. These variants differ by the functional space and the corresponding Hilbert norm selected for the function $\lambda(T)$. Certain qualitative properties of the indicated variants were discussed in [8].

In the present work, we considered the problem on determination of the nonlinear heat-conductivity coefficient of a body with an axial symmetry in the cylindrical coordinate system. (Such a problem should be solved, e.g., in the process of determining the thermodynamic properties of seamless metal pipes subjected to a high-temperature thermal treatment.) The formulation of an inverse problem and the description of the algorithmic part of the functional identification are presented. Difference schemes for nonlinear nonstationary equations, constructed by the integro-interpolation method and realized with the use of the iteration method, are constructed. Computational experiments are described and the results of calculations are discussed.

Formulation of the Problem. We will consider a nonlinear parabolic equation for the region $\Omega = \left\{(r, t): R_1 < r < R_2, 0 < t \le t_f\right\}$ of a symmetric body in the cylindrical coordinate system:

$$c(T)\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda(T)\frac{\partial T}{\partial r}\right) \tag{1}$$

[a]Institute of Mathematics, National Academy of Sciences of Belarus, 11 Surganov Str., Minsk, 220072, Belarus; [b]Presidium of the National Academy of Sciences of Belarus, 66 Nezavisimost' Ave., Minsk, 220072, Belarus; [c]Belarusian National Technical University, 65 Nezavisimost' Ave., Minsk, 220013, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 79, No. 6, pp. 23–30, November–December, 2006. Original article submitted May 30, 2006.

with the following initial and boundary conditions:

$$T(r, 0) = \bar{T}(r), \quad r \cup (R_1, R_2); \tag{2}$$

$$T(R_1, t) = g_1(t), \quad T(R_2, t) = g_2(t), \quad t \cup [0, t_f]. \tag{3}$$

These conditions are coordinated and the functions $c(T)$, $\lambda(T)$, $\bar{T}(r)$, $g_1(r)$, and $g_2(r)$ are prescribed.

Problem (1)–(3) represents a direct problem on determination of the function $T(r, t)$. Hereinafter we will consider an inverse problem, in which an unknown quantity, apart from $T(r, t)$, will be the heat-conductivity coefficient of the body. This problem is represented by problem (1)–(3) with the additional condition

$$T(r_*, t) = \tilde{T}(t), \quad t \cup [0, t_f], \tag{4}$$

where r_* is a fixed point at which the temperature of the body is measured. The point r_* lies on the segment $[R_1, R_2]$.

The above-formulated inverse problem is a classical problem (see, e.g., [2–5]). The methodical basis for the solution of this problem is provided by the known results of N. V. Muzylev, M. V. Klibanov, O. M. Alifanov, and other researches [2, 5, 9, 10], who have shown that the indicated problem can be uniquely solved in the case where additional restrictions are imposed on the initial parameters $c(T)$, $\bar{T}(r)$, $g_1(t)$, $g_2(t)$, and $\tilde{T}(t)$. In particular, in addition to the standard requirements that the function $c(T)$ must be positive, continuous, and piecewise-differentiated, for the uniqueness of the solution of the problem it will suffice to assume that $\bar{T}(r) = \text{const}$ and the quantities $g_1(t)$, $g_2(t)$, and $\tilde{T}(t)$ are monotonically increasing functions.

Computational Formulas. We now consider the functional-identification algorithm used in the inverse problem (1)–(4) and substantiated in [7]. It represents an iteration process of minimization of the goal functional (the square of the residual functional)

$$J(\lambda) = \int_0^{t_f} (T(r_*, t) - \tilde{T}(t))^2 \, dt. \tag{5}$$

It is assumed that the region G, in which the functional J is determined, coincides with any open region in the space $L_2[T^{(1)}, T^{(2)}]$ ($T^{(1)} = \min\limits_{(r,t) \cup \bar{\Omega}} T(r, t)$ and $T^{(2)} = \max\limits_{(r,t) \cup \bar{\Omega}} T(r, t)$, where $\bar{\Omega}$ is the closure of the region Ω) of square functions of the form of $\lambda(T)$ with the Hilbert norm

$$\|\lambda\| = \left(\int_{T^{(1)}}^{T^{(2)}} (\lambda(\xi))^2 \, d\xi \right)^{\frac{1}{2}}. \tag{6}$$

The algorithm of functional identification of the inverse problem (1)–(4) represents a system of recurrence relations:

$$\lambda_{n+1}(T) = \lambda_n(T) + \beta_n l_n(T), \quad n \cup \{0, 1, 2, ...\}, \tag{7}$$

$$l_n(T) = J'_{\lambda_n}(T) - \gamma_{n-1} l_{n-1}(T), \quad n \cup \{0, 1, 2, ...\}, \tag{8}$$

where $\lambda_{n+1}(T)$ is the $n+1$st approximation of $\lambda(T)$, $l_n(T)$ is the direction of descent in the $n+1$st iteration, the parameters β_n, γ_{n-1}, and J'_{λ_m} are be determined below. The initial data used for construction of the algorithm are the quantity l_{-1}: $l_{-1} = 0$; the initial approximation $\lambda_0(T)$ of the function $\lambda(T)$; the temperature at the point r_* equal to $\tilde{T}(t)$, where $t \cup [0, t_f]$; and the output data in the $n+1$st iteration — the functions $\lambda_{n+1}(T)$ and $l_n(T)$. The recurrence rela-

tions (7) and (8) represent a variant of the conjugate- gradient method [2]. An abstract scheme of this method is presented in [7]. The iteration process is terminated when the inequality

$$\int_0^{t_f} p_n(t)^2\, dt \le \varepsilon \tag{9}$$

is fulfilled for the residual

$$p_n(t) = T_n(r_*, t) - \tilde{T}(t), \tag{10}$$

where ε is a prescribed empirical parameter and $T_n(r, t)$ is the solution of the initial boundary problem

$$\begin{gathered} c(T_n)\frac{\partial T_n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda(T_n)\frac{\partial T_n}{\partial x}\right), \quad (r, t) \cup \Omega, \\ T_n(r, 0) = \bar{T}(r), \quad r \cup (R_1, R_2), \\ T_n(r_1, t) = g_1(t), \quad T_n(R_2, t) = g_2(t), \quad t \cup [0, t_f]. \end{gathered} \tag{11}$$

The parameter ε is selected on the basis of the residual and generalized-residual criteria, described in detail in [2]. Note that it is well to use the procedure of renewal of the iteration process (7), (8). This procedure involves the selection of the initial approximation in the form of $\lambda_0(T) = \lambda_n(T)$ in the case where the goal functional (5) increases in the $n + 1$st iteration.

We now determine the parameters β_n, γ_{n-1}, and J'_{λ_n}. The function J'_{λ_n} is the gradient of functional (5) at the point $\lambda = \lambda_n$, which is calculated by the formula

$$\begin{gathered} J'_{\lambda_n} = -\frac{d}{dz}\int_{R_1}^{R_2}\int_0^{t_f} \chi(z, T_n(r, t))\, r\frac{\partial T_n(r, t)}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{r}w(r, t)\right) dt dr \equiv \\ \equiv -\frac{d}{dz}\int_\Omega R_n(z, r, t)\, r\frac{\partial T_n(r, t)}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{r}w(r, t)\right) dr dt. \end{gathered} \tag{12}$$

Here, $\chi(z, s) = \begin{cases} 1, & T^{(1)} \le s \le z, \\ 0, & z < s \le T^{(2)} \end{cases}$ is the characteristic function of the set $\left\{s \,|\, T^{(1)} \le s \le z\right\}$ and $R(z, r, t)$ is the characteristic function of the set $\omega(z) = \left\{(r, t) \cup \Omega \,|\, T_n(r, t) \le z \le T^{(2)}\right\}$. The function $w = w(r, t)$ in formula (12) is a solution of the initial boundary problem

$$\begin{gathered} c(T_n)\frac{\partial w}{\partial t} + \lambda_n(T_n)\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}w\right)\right) - \delta(r - r_*)\, p_n(t) = 0, \quad (x, t) \cup \Omega, \\ w(r, t_f) = 0, \quad r \cup (R_1, R_2), \\ w(R_1, t) = w(R_2, t) = 0, \quad t \cup [0, t_f]. \end{gathered} \tag{13}$$

It should be noted that the gradient J'_{λ_n} is calculated by formula (12) on the basis of the integro-differential representations of the operators conjugate to the internal-superposition operator, determined in [7] in the cylindrical coordinate system.

The parameter γ_{n-1} is equal to the ratio between the squares of the gradient norms

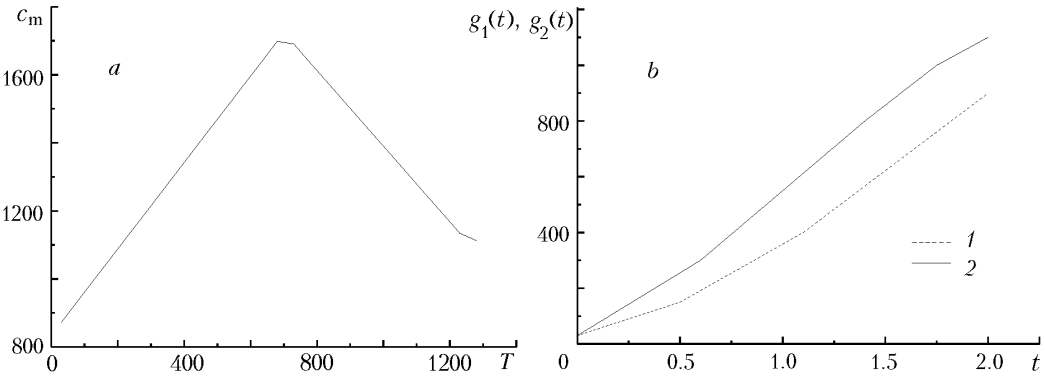

Fig. 1. Initial data of the problem: a) $c_m(T)$; b) $g_1(t)$ (1) and $g_2(t)$ (2). $c_m(T)$, W·h/(m^3.^{o}C); T, ^{o}C; $g_1(t)$, $g_2(t)$, ^{o}C; t, h.

$$\gamma_{n-1} = \frac{\|J'_{\lambda_n}\|^2}{\|J'_{\lambda_{n-1}}\|^2}, \tag{14}$$

the calculation of which with the use of (6) gives

$$\|J'_{\lambda_n}\|^2_{L_2} = \int_{T^{(1)}}^{T^{(2)}} \left(J'_{\lambda_n}\right)^2 dz . \tag{15}$$

Finally, the parameter β_n is determined as

$$\beta_n = \frac{\int_0^{t_f} p_n(s)\, \nu(r_*, s)\, ds}{\int_0^{t_f} \nu^2(r_*, s)\, ds}, \tag{16}$$

where the function $\nu = \nu(r, t)$ is a solution of the initial boundary problem

$$\frac{\partial}{\partial t}(c(T_n)\nu) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\lambda_n \nu)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r l_n(T_n)\frac{\partial T_n}{\partial r}\right), \quad (r, t) \cup \Omega ,$$

$$\nu(r, 0) = 0 , \quad r \cup (R_1, R_2) , \tag{17}$$

$$\nu(R_1, t) = \nu(R_2, t) = 0 , \quad t \cup [0, t_f] .$$

For simplicity, the subscript n is omitted for the functions w and ν in formulas (12)–(17).

Numerical Simulation. The numerical algorithm of solving the initial boundary problem (1)–(4) is constructed by the method of grids [11]. The difference schemes for different classes of nonlinear parabolic equations, used in the method of functional identificaiton, are constructed by the integro-interpolational method. The nonlinear systems of approximate algebraic equations are solved using the convergent iteration procedures developed by us. For calculating the integrals in (12), we derived formulas of the third order of accuracy.

The characteristic results of our computational experiments will be illustrated by two model examples, in which the model functions are the bilinear functions:

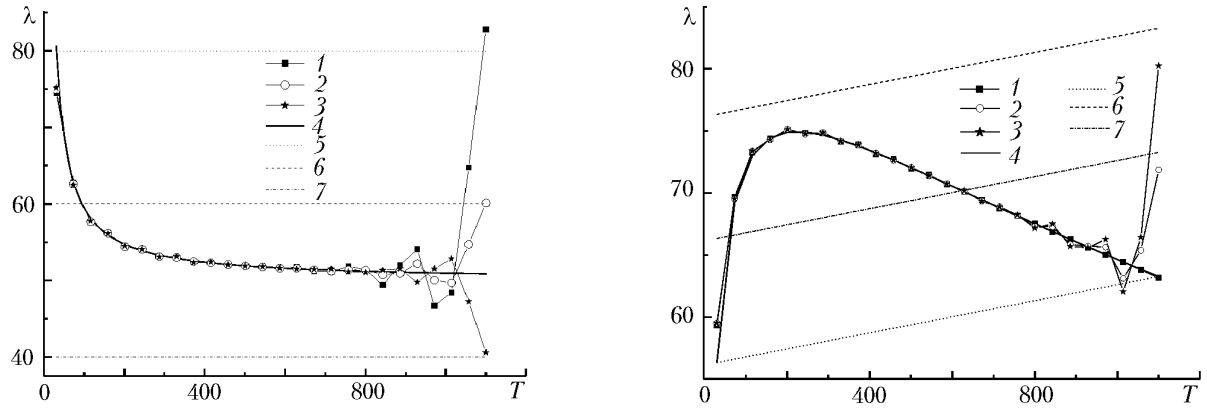

Fig. 2. Influence of the choice of the initial approximation $\lambda_0(T)$ on the result of identification of the heat-conductivity coefficient $\lambda_{\mathrm{m}}^{(1)}(T)$: 1) $\lambda_{n_1}^{(1)}$ ($\lambda_0 = \lambda_0^1$); 2) $\lambda_{n_2}^{(1)}$ ($\lambda_0 = \lambda_0^2$); 3) $\lambda_{n_3}^{(1)}$ ($\lambda_0 = \lambda_0^3$); 4) $\lambda_{\mathrm{m}}^{(1)}(T)$; 5) $\lambda_0 = \lambda_0^1$; 6) $\lambda_0 = \lambda_0^2$; 7) $\lambda_0 = \lambda_0^3$. λ, W/(m^3.°C); T, °C.

Fig. 3. Dependence of the results of identification of the heat-conductivity coefficient $\lambda_{\mathrm{m}}^{(2)}(T)$ on the choice of the initial approximation $\lambda_0(T)$: 1) $\lambda_{n_1}^{(2)}$ ($\lambda_0 = \lambda_0^1$); 2) $\lambda_{n_2}^{(2)}$ ($\lambda_0 = \lambda_0^2$); 3) $\lambda_{n_3}^{(2)}$ ($\lambda_0 = \lambda_0^3$); 4) $\lambda_{\mathrm{m}}^{(2)}(T)$; 5) $\lambda_0 = \lambda_0^1$; 6) $\lambda_0 = \lambda_0^2$; 7) $\lambda_0 = \lambda_0^3$. λ, W/(m^3.°C); T, °C.

$$\lambda_{\mathrm{m}}^{(1)}(T) = \frac{50.0T + 1000.0}{T + 1.0}, \tag{18}$$

$$\lambda_{\mathrm{m}}^{(2)}(T) = \frac{85.0T + 0.4}{0.0003T^2 + T + 15.0}. \tag{19}$$

The functions $c_{\mathrm{m}}(T) = c(T)$, $g_1(t)$, and $g_2(t)$ are presented in Fig. 1a and b respectively.

It is assumed that $R_1 = 0.05$, $R_2 = 0.3$, $t_{\mathrm{f}} = 2.0$, and $T(r) = 30.0$. The function $\widetilde{T}$ is a numerical solution of problem (1)–(3) at the point $r = r_*$ on condition that $c(T) = c_{\mathrm{m}}(T)$ and $\lambda(T) = \lambda_{\mathrm{m}}^{(i)}(T)$, where $i = 1, 2$. This solution was obtained using grids with fairly small space and time pitches, a decrease in which has practically no influence on the final result. For both model examples presented below, the numerical differentiation in (12) was performed by the formulas of the first order of approximation [11] because, as was established in the process of numerical experiments, the use of formulas of higher orders of approximation, as high as the second one, does not increase the accuracy of the calculations.

The results of different computational experiments show that the calculated value of the heat conductivity coefficient $\lambda(T)$ of a body depends substantially not only on the temperature of the body, but also on the choice of the initial approximation $\lambda_0(T)$, the position of the point r_* at which the temperature is measured, the size of the grid pitches, and the value of ε in relation (9). For concrete problems, it is difficult to a priori determine even one value of these parameters providing the best result in the ratio between the accuracy and rate of the calculations. This situation is characteristic of the nonlinear numerical analysis as a whole. The influence of the above-indicated factors on the accuracy and rate of the calculations is complex in character, which is evident from the graphs presented in the figures.

Figures 2 and 3 illustrate the influence of the choice of the initial approximation on the estimated values of the coefficients $\lambda_{\mathrm{m}}^{(i)}(T)$, where $i = 1, 2$. The calculations were carried out using a grid with a space pitch $h = 0.005$ and a time pitch $\tau_t = 0.01$ at $r_* = 0.2$ and $\varepsilon = 0.0001$. It is seen from Fig. 2 that the quantities $\lambda_n(T\big|_{T=T^{(1)}+0})$ and $\lambda_n(T\big|_{T=T^{(2)}-0})$ are close to the quantities $\lambda_0(T^{(1)})$ and $\lambda_0(T^{(2)})$ respectively. Consequently, the accuracy of identification

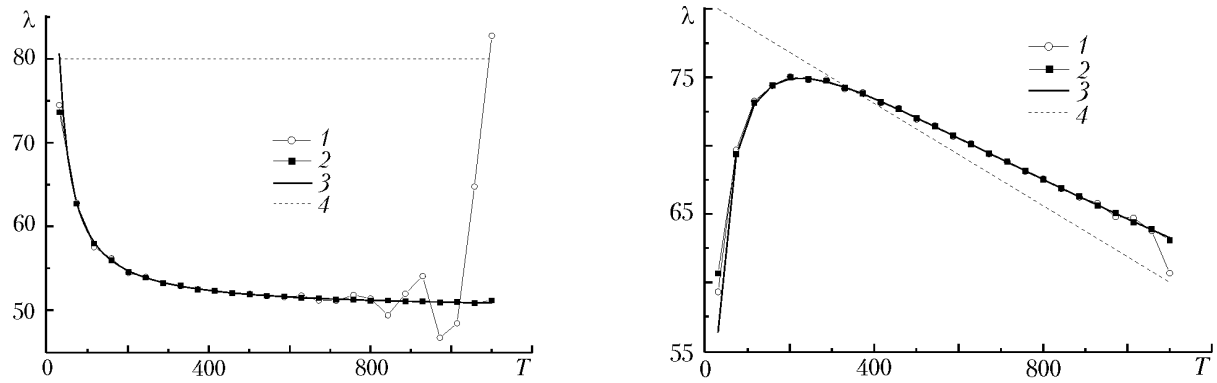

Fig. 4. Results of calculating the coefficient $\lambda_m^{(1)}(T)$ depending on the value of r_*; 1) $\lambda_{n_1}^{(1)}$ (r_* = 0.2); 2) $\lambda_{n_2}^{(1)}$ (r_* = 0.275); 3) $\lambda_m^{(1)}(T)$; 4) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C.

Fig. 5. Dependence of the results of calculating the coefficient $\lambda_m^{(2)}(T)$ on the value of r_*: 1) $\lambda_{n_1}^{(2)}$ (r_* = 0.2); 2) $\lambda_{n_2}^{(1)}$ (r_* = 0.275); 3) $\lambda_m^{(2)}(T)$; 4) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C.

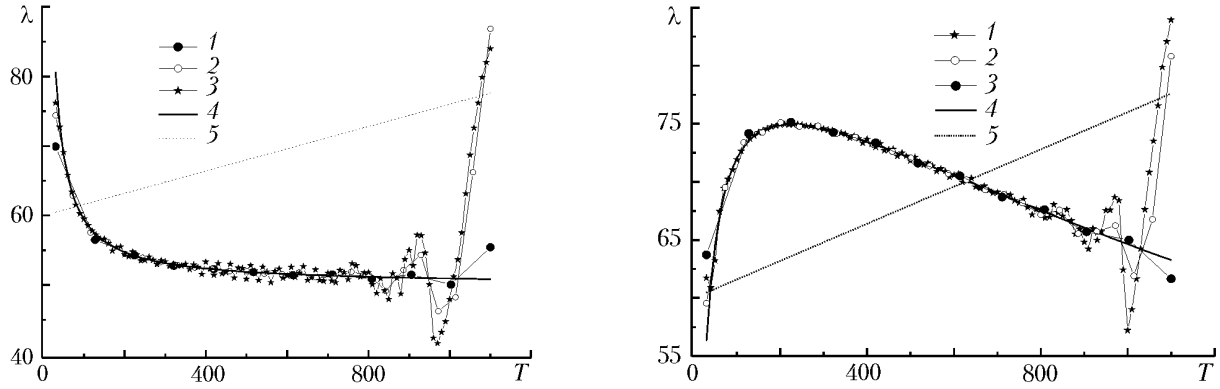

Fig. 6. Results of estimating the coefficient $\lambda_m^{(1)}(T)$ depending on the steps of numerical differentiation of the operator in (12): 1) $\lambda_{n_1}^{(1)}$ (τ_T = 10.0); 2) $\lambda_{n_2}^{(1)}$ (τ_T = 42.8); 3) $\lambda_{n_3}^{(1)}$ (τ_T = 97.0); 4) $\lambda_m^{(1)}(T)$; 5) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C.

Fig. 7. Influence of the steps of numerical differentiation of the operator in (12) on the result of estimating the coefficient $\lambda_m^{(2)}(T)$: 1) $\lambda_{n_1}^{(2)}$ (τ_T = 10.0); 2) $\lambda_{n_2}^{(2)}$ (τ_T = 42.8); 3) $\lambda_{n_3}^{(2)}$ (τ_T = 97.0); 4) $\lambda_m^{(2)}(T)$; 5) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C.

of $\lambda(T)$ at the ends of the segment $[T^{(1)}, T^{(2)}]$ can be improved by selecting an initial approximation that at the ends of this segment would be identical or fairly close to $\lambda(T)$ at these ends (curves 2 and 3 in Fig. 2 and curve 1 in Fig. 3).

The influence of the value of r_* on the accuracy of the calculations is demonstrated in Figs. 4 and 5. The accuracy of identification of the coefficient $\lambda(T)$ at the points $T^{(1)}$ and $T^{(2)}$ depends on the distance from the point r_* to these points.

However, in the general case, for bodies in both the cylindrical and Cartesian coordinate systems [7, 8] it is difficult to estimate the values of $\lambda(T)$ at the ends of the segment $[T^{(1)}, T^{(2)}]$ with the use of the above-described algorithm. The quantities $\lambda_n(T\big|_{T=T^{(1)}+0})$ and $\lambda_n(T\big|_{T=T^{(2)}-0})$, where n = 1, 2, ..., remain close to the quantities $\lambda_0(T^{(1)})$ and $\lambda_0(T^{(2)})$ respectively. We failed to obtain satisfactory ratios between the values of $\lambda_n(T^{(1)}) \approx \lambda(T^{(1)})$ and $\lambda_n(T^{(2)}) \approx \lambda(T^{(2)})$ (Figs. 2–8) by varying the value of the grid pitches and the value of r_*.

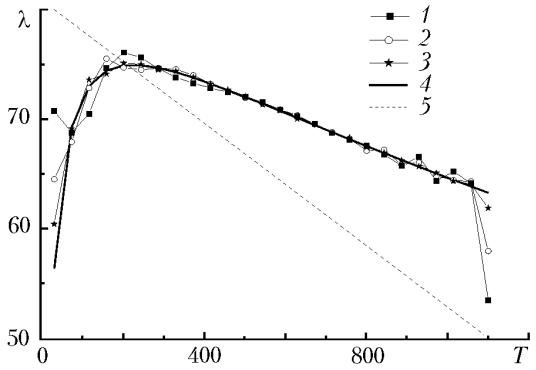

Fig. 8. Results of identification of the heat-conductivity coefficient depending on the value of the residual ε: 1) λ_{n_1} (ε = 0.1); 2) λ_{n_2} (ε = 0.01); 3) λ_{n_3} (ε = 0.001); 4) $\lambda_m^{(2)}(T)$; 5) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C.

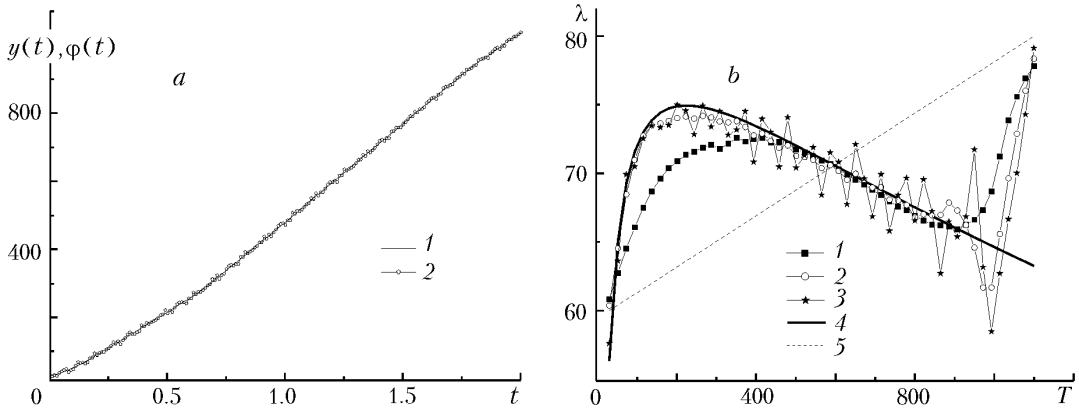

Fig. 9. Exact $y(t)$ (1) and disturbed $\varphi(t)$ (2) values of the initial data (a) and their influence on the results of calculating the coefficient $\lambda_m^{(2)}(T)$ (b): 1) λ_2; 2) λ_{13}; 3) λ_{34}; 4) $\lambda_m^{(2)}(T)$; 5) $\lambda_0(T)$. λ, W/(m^3·°C); T, °C; $y(t)$, $\varphi(t)$, °C; t, h.

Experiments carried out for different parameters have shown that the accuracy of determining $\lambda(T)$ increases when the space and time pitches are decreased to a certain value. In the case were grids with an insufficiently small pitches are used, large errors can be introduced into the calculation results. However, the use of grids with a very small pitch τ_T for numerically calculating the derivative in gradient (12) can lead to the appearance of nonphysical fluctuations and large errors in the calculations (curves 2 and 3 in Fig. 6 and curves 1 and 2 in Fig. 7). To obtain a reliable result, it is necessary to change the value of the pitch τ_T in a wide range in the process of estimating the heat-conductivity coefficient of a body.

A decrease in the value of ε in (9) leads to an increase in the accuracy of the calculations. Figure 8 shows the influence of the value of ε on the result of estimation of the coefficient $\lambda_m^{(2)}(T)$. In this case, account must be taken of the fact that the use of a very small ε can lead to a significant increase in the number of iteration cycles in algorithm (7), (8), which does not aid markedly in increasing the accuracy of the calculations. Therefore, it makes sense to determine the rate of change in the residual and to stop the calculations at a certain number of iterations. Note that the choice of ε is determined by the forms of λ_m, the initial approximation λ_0, and the value of $r = r_*$. If the initial approximation is selected inadequately, the residual can increase in the $n+1$st iteration. In this case, it is necessary to change the initial approximation or to perform the renewal procedure, in which the quantity $\lambda_n(T)$ is used as the initial approximation λ_0. Then problem (1)–(4) is solved with this initial approximation.

The results of investigation of the influence of the initial data containing a "noise" on the solution of the inverse problem on the heat conductivity of a body in the cylindrical coordinate system (Fig. 9) allows the conclusion that the solution of this problem is numerically unstable relative to the indicated data and that the number of iterations

at which the calculations by the algorithm should be stopped is determined, as in [8], by the level of the "noise." Figure 9b shows that the solution of the problem being considered is numerically unstable in the case where the initial data include high-frequency quasi-periodic fluctuations: $\varphi(t) = \tilde{T}(t) + \sum_{i=1}^{n} a_i \sin(\omega t)$ (Fig. 9a).

Conclusions. Our calculations have shown that the method proposed for functional identification of the heat-conductivity coefficient of a body and the numerical algorithms used for this purpose are fairly efficient. The indicated method allows one to determine the heat-conductivity coefficient of a body fairly exactly. However, to obtain reliable results, it is necessary to use different computational parameters in the process of calculations. It is well to visually observe, with the use of a computer, the change in the residual and the number of iterations. The algorithm developed by us can be effectively used for determining the thermophysical properties of tubular bodies.

The indicated algorithm also allows one to estimate the heat-conductivity coefficient of a body with a discontinuous derivative with respect to the temperature, which is of particular importance for estimating the heat-conductivity coefficients in transfer processes with phase transformations.

NOTATION

a, amplitude; r, radius, m; $c(T)$, heat capacity, W·h/(m^3·^{o}C); t, current instant of time, h; t_f, final instant of time, h; T, temperature, ^{o}C; β_n, descent coefficient; ε_n, residual; $\lambda(T)$, heat conductivity coefficient, W/(m·^{o}C); λ_0, zero approximation; λ_n, nth approximation; λ_m, model function; $\varphi(t)$, initial data with a "noise", ^{o}C; Ω, computational region. Subscripts: m, model; f, finite; n, number of an algorithm iteration.

REFERENCES

1. O. M. Alifanov, *Inverse Heat-Transfer Problems* [in Russian], Mashinostroenie, Moscow (1988).
2. O. M. Alifanov, E. A. Artyukhin, and S. V. Rumyantsev, *Extreme Methods of Solving Ill-Posed Problems* [in Russian], Nauka, Moscow (1988).
3. S. V. Rumyantsev, Methods of taking into account *a priori* information in regularizing gradient algorithms, *Inzh.-Fiz. Zh.*, **49**, No. 6, 932–935 (1985).
4. E. A. Artyukhin, Estimation of the thermal conductivity coefficient as a result of the solution of a nonlinear inverse problem, *Inzh.-Fiz. Zh.*, **41**, No. 4, 587–592 (1981).
5. M. N. Ozisik and B. H. R. Orlande, *Inverse Heat Transfer: Fundamentals and Applications*, Taylor and Francis, New York (2000).
6. J. Wang, A. J. Silva Neto, F. D. Moura Neto, and J. Su, Function estimation with Alifanov's iterative regularization method in linear and nonlinear heat conduction, *Appl. Math. Model.*, **26**, Issue 11, 1093–1111 (2002).
7. V. T. Borukhov and V. I. Timoshpol'skii, Functional identification of the nonlinear thermal-conductivity coefficient of a body by gradient methods. I. Conjugate operators, *Inzh.-Fiz. Zh.*, **78**, No. 4, 68–74 (2005).
8. V. T. Borukhov, V. I. Timoshpol'skii, G. M. Zayats, and V. A. Tsurko, Functional identification of the nonlinear thermal-conductivity coefficient by gradient methods. II. Numerical modeling, *Inzh.-Fiz. Zh.*, **78**, No. 4, 75–81 (2005).
9. J. R. Cannon and P. DuChateau, An inverse problem for a nonlinear diffusion equation, *SIAM J. Appl. Math.*, **39**, Issue 2, 272–289 (1980).
10. A. Lorenzi, An inverse problem for semilinear parabolic equation, *Ann. Mat. Pura Appl.,* **31**, 145–166 (1982).
11. A. A. Samarskii, *The Theory of Difference Schemes* [in Russian], Nauka, Moscow (1977).